

Chromatic weight systems and the corresponding knot invariants

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February 9, 2008

Abstract

We prove that chromatic weight systems, introduced by Chmutov, Duzhin and Lando, can be expressed in terms of weight systems associated with direct sums of the Lie algebras \mathfrak{gl}_n and \mathfrak{so}_n . As a consequence the Vassiliev invariants of knots corresponding to the chromatic weight systems distinguish exactly the same knots as a one variable specialisation Υ of the Homfly and Kauffman polynomial.

Introduction

Vassiliev invariants of knots in \mathbb{R}^3 are in one-to-one correspondence with weight systems, that is with linear forms on a vector space spanned by so-called chord diagrams. On one hand side, there exists a construction of weight systems using reductive Lie algebras. On the other hand side, Chmutov, Duzhin and Lando introduced chromatic weight systems without using Lie algebras. This paper will give a positive and complete answer to the following natural question:

- Is there a relation between chromatic weight systems and weight systems associated to Lie algebras ?

We will also study the implications of this relationship on the level of weight systems and on the level of knot invariants.

The paper is organized as follows. From Section 1 to Section 6 we prove that chromatic weight systems are linear combinations of weight systems associated with Lie algebras. For the proof we express a weight system called $W_{r,\underline{x}}$ in two different ways:

- The map $W_{r,\underline{x}}$ appears as the top coefficient of a polynomial expression for weight systems coming from the Lie algebras $\mathfrak{gl}_n^{\oplus r}$ (or $\mathfrak{so}_n^{\oplus r}$) and the r -th tensor power of the standard representation. This establishes the connection to Lie algebras.
- The map $W_{r,\underline{x}}$ can be expressed as a sum over colorings of the intersection graph of a chord diagram using r colors. This leads to the connection with chromatic weight systems.

In Section 7 we derive from the formula for $W_{r,\underline{x}}$ a formula for the value of the universal chromatic weight system in terms of embeddings of certain trivalent diagrams into an oriented 2-sphere. In Section 8 we prove that chromatic weight systems are *not* linear combinations of weight systems coming from direct sums of exceptional Lie algebras and certain Lie superalgebras which completes our results on the level of weight systems.

From Section 9 to Section 12 we translate the results from weight systems to Vassiliev invariants. We recall the relation between a weight system associated to \mathfrak{gl}_n and the Homfly polynomial H . Then we extract from H the part Υ belonging to a certain chromatic weight system W . The polynomial Υ turns out to distinguish the same knots as all Vassiliev invariants coming from chromatic weight systems. It is also a specialization of the Kauffman polynomial.

By [Vo2] there exist weight systems that do not come from semisimple Lie superalgebras. This was an inspiration for our research, as well as the need of a good combinatorial understanding of the known weight systems.

Acknowledgement

The starting-point for this work was in Bonn where I was guest at the 'Max-Planck-Institut für Mathematik' and where C. F. Bödigheimer told me about chromatic weight systems. Most of the work for this paper was carried out at the Université Louis Pasteur at Strasbourg. I would like to express my deep gratitude to C. Kassel for his help, especially for giving me important literature references and for proposing many small and great changes in earlier versions of this paper. I thank the German Academic Exchange Service for financial support (Doktorandenstipendium HSP II/AUFE).

1 The bialgebra of chord diagrams

Consider finite unoriented graphs D (multiple edges are allowed) with the following properties:

- 1) D is equipped with an embedding of an oriented circle $S^1 \subset D$.
- 2) Every vertex has valency three.
- 3) The vertices outside of the image of S^1 are equipped with a cyclic order on the three arriving edges.

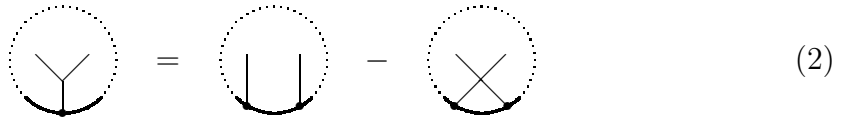
Graphs with these properties are said to be equivalent if there exists a homeomorphism of graphs that respects the embeddings of S^1 with orientation and preserves the cyclic order of the vertices outside of S^1 . An equivalence class of these graphs will be called a *trivalent diagram*. A trivalent diagram D will be called a *chord diagram* if all vertices lie on $S^1 \subset D$. We define the degree of D by

$$\deg D := \frac{1}{2} \# \{\text{trivalent vertices of } D\} \in \mathbb{N}. \quad (1)$$

In our pictures of trivalent diagrams the embedded oriented circle S^1 is drawn with a thick line and is always oriented counterclockwise. The rest of the diagram is drawn in the interior of the disk bounded by S^1 . The cyclic order of the trivalent vertices outside of S^1 is counterclockwise. Vertices with valency four in our pictures do not correspond to vertices of the graph. They only appear because we draw the diagrams in the plane. We use dots to indicate that we only show a part of a trivalent diagram.


Let R be a field of characteristic 0. The symbol \otimes will always denote the tensor product over R . If B is a graded R -vector space, we will denote the space of homogeneous elements of degree d by B_d and the space spanned by all homogeneous elements of degree $\leq d$ ($\geq d$) by $B_{\leq d}$ ($B_{\geq d}$).

Definition 1.1 *Let A be the graded R -vector space with trivalent graphs as generators and with the so-called (STU)-relation from Picture (2) for all triples of diagrams that only differ like the three diagrams in this picture.*



$$\text{(Diagram 1)} = \text{(Diagram 2)} - \text{(Diagram 3)} \quad (2)$$

The implicit statement in Definition 1.1 that A is a graded vector space holds because the (STU)-relation is homogeneous with respect to the degree of trivalent diagrams. The (STU)-relation allows one to write every element of A as a linear combination of chord diagrams. Applying the (STU)-relation two times gives the so called (4T)-relation (= four-term relation) from Picture (3) for chord diagrams.



$$(3)$$

As a consequence there exists a surjective morphism from the vector space generated by chord diagrams and (4T)-relations to A induced by the inclusion of sets of diagrams. This map is an isomorphism (see [BN1], Theorem 6).

Now we describe the Hopf algebra structure of A . Given two trivalent diagrams D_1 and D_2 , cut their oriented circles somewhere outside of a vertex and glue the resulting oriented intervals to form one new oriented circle. This operation will be called the connected sum of D_1 and D_2 . It can be proved that the connected sum is well defined for classes of diagrams in A , so it turns A into a commutative graded algebra with unit S^1 .

For a trivalent diagram D define

$$\Delta_r(D) := \sum_{D=D_1 \cup \dots \cup D_r} D_1 \otimes \dots \otimes D_r \quad (4)$$

where in the case of $\deg D > 0$ the sum is taken over all functions $\varphi : \pi_0(D \setminus S^1) \longrightarrow \{1, \dots, r\}$ and the diagrams D_1, \dots, D_r in the sum are defined by

$$D_i = D \setminus \bigcup_{E: \varphi(E) \neq i} E.$$

For $D = S^1$ the notation in Formula (4) will mean that $\Delta_r(S^1) = S^1 \otimes \dots \otimes S^1$.

The map $\Delta := \Delta_2$ turns A into a commutative, cocommutative bialgebra and we have $\Delta_r = (\Delta_{r-1} \otimes id) \circ \Delta$. The counit ϵ is determined by

$$\epsilon(D) = \begin{cases} 1 & \text{if } D = S^1, \\ 0 & \text{if } \deg D > 0. \end{cases} \quad (5)$$

We also have $\dim A_0 = 1$ and $\dim A_i < \infty$ for all i . A structure theorem ([MiM]) implies that the graded bialgebra A is a Hopf algebra isomorphic to the algebra of polynomials on the graded vector space

$$P(A) := \{a \in A \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}$$

of primitive elements of A . By the definition of Δ it is clear that

$$\mathcal{M} := \left\{ \begin{array}{l} \text{trivalent diagrams } D \text{ such that} \\ D \setminus S^1 \text{ is connected} \end{array} \right\} \subset P(A). \quad (6)$$

On the other hand, the (STU)-relation allows one to write every trivalent diagram as a linear combination of products of elements from \mathcal{M} . This implies that $\text{span } \mathcal{M} = P(A)$. We do not need the so-called Chinese characters (see [BN1]) for this purpose. Useful applications of these diagrams different to the ones in [BN1] are a decomposition of A_d into a direct sum of eigenspaces of the cabling operations ([KSA]) and a proof that slight modifications of the definition of $P(A)$ are possible (Proposition 1.1 of [Vo2]). We will use the last application implicitly in Section 8.

2 Weight systems

An open question concerning the structure of A is the determination of (the asymptotic behaviour of) the sequence $(\dim P(A)_i)_{i \geq 10}$. The best we can do in higher degrees in order to obtain some information from a linear combination of classes of trivalent diagrams in A is to apply an element of the dual space of A to this linear combination.

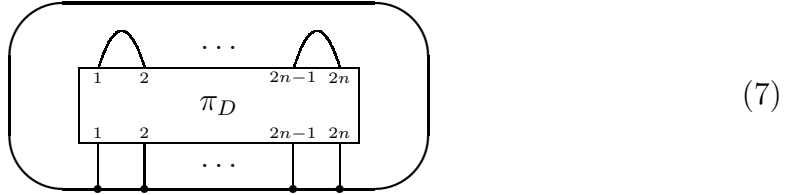
Definition 2.1 *Call a linear map $A_d \longrightarrow R$ a weight system of degree d , a linear map $A \longrightarrow R$ a weight system, and an algebra homomorphism $A \longrightarrow R$ a multiplicative weight system.*

The dual space $A^* \cong \prod_{i=0}^{\infty} A_i^*$ of all weight systems is an algebra and the graded dual space $\bigoplus_{i=0}^{\infty} A_i^*$ generated by all weight systems of finite degree is a graded Hopf algebra. The algebra A^* becomes a Hopf algebra in a category with a completed tensor product and the multiplicative weight systems are the group-like elements of this Hopf algebra. Instead of R we may take some R -vector space M as the space of values of weight systems. Composition with elements of M^* gives us weight systems with values in R . A weight system w with values in a finite dimensional vector space M is said to be a linear combination of other weight systems with values in R if this is true for $p \circ w$ for all $p \in M^*$.

Weight systems can be obtained from Lie superalgebras. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}/(2)$ -graded vector space with a graded bilinear map $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$

satisfying an antisymmetry property and a Jacoby identity. Lie algebras are Lie superalgebras with $\mathfrak{g}_{\bar{1}} = \{0\}$. We will consider the case $\mathfrak{g}_{\bar{1}} \neq \{0\}$ only in this section and in Section 8. Let $(\ , \)$ be a consistent, supersymmetric ad-invariant, non-degenerate bilinear form on the Lie superalgebra \mathfrak{g} (see [Kac] for the terminology). An example of such a bilinear form is the Killing form for a semisimple Lie algebra. We choose two homogeneous bases e_i and e'_j of \mathfrak{g} such that $(e'_j, e_i) = \delta_{ij}$ ($i, j = 1, \dots, \dim \mathfrak{g}$) and define $\omega := \sum_i e_i \otimes e'_i$. The definition of ω does not depend on the choice of the bases and ω is supersymmetric, ad-invariant and of degree 0. We call ω a Casimir element. Let $\rho : \mathfrak{g} \longrightarrow \text{End } M$ be a representation of \mathfrak{g} in a $\mathbb{Z}/(2)$ -graded, finite-dimensional vector space M . To $(\rho \otimes \rho)(\omega)$ we will associate a weight system as follows:

Let D be a chord diagram of degree n . Then D looks as in Picture (7) for a non-unique permutation $\pi_D \in S_{2n}$ where in the box named π_D the top chord-end number i is connected to the bottom chord-end number $\pi_D(i)$.



The symmetric group S_{2n} acts on $(\text{End } M)^{\otimes 2n}$ from the left by superpermutation of the tensor factors:

$$\pi \cdot \varphi_1 \otimes \dots \otimes \varphi_{2n} := (-1)^{\sigma(\pi, \deg \varphi_1, \dots, \deg \varphi_{2n})} \varphi_{\pi^{-1}(1)} \otimes \dots \otimes \varphi_{\pi^{-1}(2n)} \quad (8)$$

where

$$\sigma(\pi, a_1, \dots, a_{2n}) := \# \{ (i, j) \mid 1 \leq i < j \leq 2n, \pi(i) > \pi(j), a_i = a_j = \bar{1} \}.$$

Let $\gamma_{2n} : \text{End } M^{\otimes 2n} \longrightarrow \text{End } M$ be the composition of $2n$ endomorphisms and define

$$W_{\mathfrak{g}, \omega, \rho}(D) := \begin{cases} \text{str}(\gamma_{2n}((\pi_D \cdot (\rho^{\otimes 2n}(\omega^{\otimes n})))))) & \text{if } \deg D > 0, \\ \text{sdim } M := \dim M_{\bar{0}} - \dim M_{\bar{1}} & \text{if } D = S^1 \end{cases} \quad (9)$$

where str denotes the supertrace.

Lemma 2.1 *Formula (9) defines a weight system $W_{\mathfrak{g}, \omega, \rho} : A \longrightarrow R$.*

For more useful details, extensions or generalisations of this construction of weight systems and for a proof of Lemma 2.1 see Section 6 of [Vo2], [Vai] and Section 2.4 of [BN1].

Observe that for $a \in R$ we have

$$W_{\mathfrak{g}, a\omega, \rho}(D) = a^{\deg D} W_{\mathfrak{g}, \omega, \rho}(D). \quad (10)$$

Define a contraction map $\kappa_{2n} : (M \otimes M^*)^{\otimes 2n} \longrightarrow R$ by

$$\kappa_l(m_1 \otimes \varphi_2 \otimes m_2 \otimes \dots \otimes \varphi_{2n} \otimes m_{2n} \otimes \varphi_1) := (-1)^{\deg m_1} \prod_{i=1}^{2n} \varphi_i(m_i). \quad (11)$$

Let $\iota_M : \text{End } M \longrightarrow M \otimes M^*$ be the canonical isomorphism. Then we can restate Formula (9) for $\deg D > 0$ as

$$W_{\mathfrak{g}, \omega, \rho}(D) = (\kappa_{2n} \circ \iota_M^{\otimes 2n}) (\pi_D \cdot (\rho^{\otimes 2n}(\omega^{\otimes n}))). \quad (12)$$

3 Chromatic weight systems

Now we review some facts from [CDL2] and [CDL3] concerning chromatic weight systems. Consider a chord diagram D . We call $\pi_0(D \setminus S^1)$ the set of chords of D . We say that the chords a and b intersect if we meet the endpoints of a and b in alternating order when we travel around the circle of D .

Definition 3.1 *The intersection graph $\Gamma(D)$ of a chord diagram D is a graph whose vertices are in bijection with the chords of D and two vertices are connected by an edge exactly if the corresponding chords of D intersect.*

Now consider graphs G without orientation, without loops and without multiple edges and let $M_0(G)$ denote the set of vertices of G . A weighted graph G is a graph together with a function $w : M_0(G) \longrightarrow \mathbb{N}$. Define the degree of G as the sum of the weights of all vertices. Given an edge $e =]v_1; v_2[$ of a weighted graph G , we can define two new weighted graphs G'_e and G''_e :

- G'_e is obtained from G by removing the edge e .
- G''_e is obtained from G'_e by first contracting the vertices v_1 and v_2 to one new vertex with the weight $w(v_1) + w(v_2)$ and then by replacing multiple edges by unique edges.

Definition 3.2 Let \mathcal{C} be the graded R -vector space with weighted graphs including the empty graph as generators and with the 'chromatic' relations $G = G'_e - G''_e$ for all edges e .¹

We have a map from chord diagrams to weighted graphs such that a chord diagram D is mapped to $\Gamma(D)$ with the weight of all vertices being one. This map induces a linear map

$$\bar{\Gamma} : A \longrightarrow \mathcal{C} \quad (13)$$

of graded modules and there exists a structure of a Hopf algebra on \mathcal{C} that turns this map into a morphism of Hopf algebras. The product of two weighted graphs is their disjoint union and the coproduct is described in [CDL3]. A consequence of the chromatic relations is that \mathcal{C} is spanned by products of single vertices where the weights may be chosen freely. This means that there exists a surjective morphism of graded algebras

$$R[s_1, s_2, \dots] \longrightarrow \mathcal{C} \quad , \quad s_d \mapsto \underset{\bullet}{d} \quad , \quad (\deg s_d := d) \quad (14)$$

sending s_d to the graph consisting of a single vertex with weight d . By Theorem 2 of [CDL3] this map is an isomorphism.

Definition 3.3 A weight system that factors through \mathcal{C} will be called chromatic and $\bar{\Gamma}$ will be called the universal chromatic weight system.

The space of chromatic weight systems is a subalgebra of A^* isomorphic to \mathcal{C}^* because $\bar{\Gamma}$ is a surjective morphism of coalgebras.

We have described two types of weight systems that are not obviously related to each other. One of our main goals is to prove that chromatic weight systems come from Lie algebras.

Theorem 3.1 Every chromatic weight system of degree d is a linear combination of weight systems associated with the Lie algebras $\mathfrak{gl}_n^{\oplus d}$ and also a linear combination of weight systems associated with the Lie algebras $\mathfrak{so}_n^{\oplus d}$.

For the previous theorem we can use for both Lie algebras the d -th tensor power of the defining representation. The Casimir element and n vary for the weight systems used in the linear combination. We prove Theorem 3.1 in Section 6.

¹The relations $G = G'_e + G''_e$ of [CDL3] lead (up to signs) to the same, but would cause slightly more complicated formulas here. The chromatic relation should be compared to the (STU)-relation.

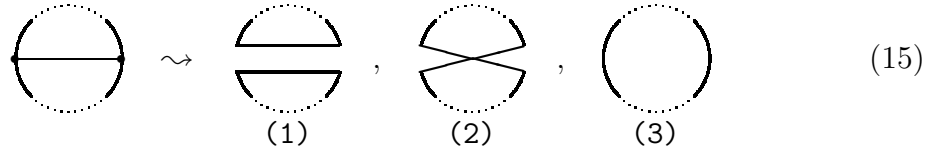
4 Weight systems coming from \mathfrak{gl}_n

For a chord diagram D define

$$W(D) := \begin{cases} 0 & \text{if } D \text{ has two intersecting chords,} \\ 1 & \text{otherwise.} \end{cases}$$

It will follow from Lemma 4.2 and is easy to see directly that W induces a multiplicative weight system $W : A \longrightarrow R$. In each finite degree we want to obtain W as a linear combination of weight systems coming from the Lie algebras \mathfrak{gl}_n and \mathfrak{so}_n . We start with a purely combinatorial lemma. It is formulated so as to be useful for the proofs of Lemma 4.2, Lemma 4.3, and also for a possible version of Lemma 4.3 for the Lie algebra \mathfrak{sl}_n (see Exercise 6.33 and Exercise 6.34 of [BN1] or write down an explicit formula for a Casimir element of \mathfrak{sl}_n).

Lemma 4.1 *Let D be a chord diagram with d chords. Forget the orientation of S^1 . If we replace each chord by one of the three possibilities from Picture (15), then we get $k \leq d + 1$ circles. We get $d + 1$ circles exactly if no chords of D intersect and every chord is replaced by Possibility (1) of Picture (15).*



Proof: The lemma can be proved by induction on the number of chords using chord diagrams on some number of circles. \square

We recall the graphical calculus for a weight system associated to \mathfrak{gl}_n from [BN1]. Let M be an n -dimensional vector space with basis e_i ($i = 1, \dots, n$). Let e_{ij} denote the standard basis of $\text{End } M$ and let $\tau := \text{id} : \mathfrak{gl}_n \longrightarrow \text{End } M$ be the defining representation of the Lie algebra \mathfrak{gl}_n . Define the ad-invariant, symmetric, non-degenerate bilinear form $(\ , \)$ by $(a, b) := \text{tr}(\tau(a)\tau(b))$. The corresponding Casimir element is

$$X = \sum_{i,j=1}^n e_{ij} \otimes e_{ji} \in \mathfrak{gl}_n \otimes \mathfrak{gl}_n. \quad (16)$$

As a linear map X is the permutation of the tensor factors of $M \otimes M$. Now recall the expression $W_{\mathfrak{gl}_n, X, \tau}$ from Formula (12). Let e^i be the basis of M^* dual to e_i ($i = 1, \dots, n$). Then

$$(\iota_M \otimes \iota_M) \circ (\tau \otimes \tau)(X) = \sum_{i,j=1}^n e_i \otimes e^j \otimes e_j \otimes e^i. \quad (17)$$

This tensor appears in a 'big sum expansion' of $W_{\mathfrak{gl}_n, X, \tau}(D)$ with $n^{4 \deg D}$ terms as indicated by Picture (18).


(18)

A term with two vectors e_{i_ν} and e^{i_μ} that label an interval on the oriented circle between two neighbored chord-ends can only give a contribution to the 'big sum' if $i_\nu = i_\mu$. This and Picture (18) implies that

$$W_{\mathfrak{gl}_n, X, \tau}(D) = n^{c(D)} \quad (19)$$

where $c(D)$ denotes the number of circles that appear when all chords of D are replaced by Possibility (1) of Picture (15). We summarize:

Lemma 4.2 *The formula*

$$W_{\mathfrak{gl}}(D) = t^{c(D)-1}$$

for chord diagrams D defines a homomorphism of algebras $W_{\mathfrak{gl}} : A \longrightarrow R[t]$ with the property

$$W_{\mathfrak{gl}_n, X, \tau}(D) = n W_{\mathfrak{gl}}(D)(n)$$

for all $n \geq 1$. If D has degree d , then $\deg(W_{\mathfrak{gl}}(D)) \leq d$ and the coefficient of t^d is $W(D)$.²

²The constant term in $W_{\mathfrak{gl}}$ is the weight system of the Alexander-Conway polynomial and $W_{\mathfrak{gl}}$ can also be seen to be well-defined for geometric reasons (see [BNG], Section 3).

Proof: The map $W_{\mathfrak{gl}}$ is well defined because in Formula (19) infinitely many choices of n are possible. It is multiplicative with respect to the connected sum of chord diagrams because c is additive with respect to the disjoint union of chord diagrams. By Lemma 4.1 we have $c(D) \leq d + 1$ for all chord diagrams D and $c(D) = d + 1$ exactly if no chords of D intersect. This completes the proof. \square

We can formulate a version of Lemma 4.2 also for the Lie algebras \mathfrak{so}_n .

Lemma 4.3 *Let τ be the defining representation of \mathfrak{so}_n . We use a bilinear form $(\ , \)$ defined by $(a, b) := \frac{1}{2} \text{tr}(\tau(a)\tau(b))$ and denote the associated Casimir element by ω . Then there exists a morphism of R -algebras $W_{\mathfrak{so}} : A \longrightarrow R[t]$ such that for all $n \geq 2$ we have*

$$W_{\mathfrak{so}_n, \omega, \tau}(D) = nW_{\mathfrak{so}}(D)(n).$$

If D has degree d , then $\deg(W_{\mathfrak{so}}(D)) \leq d$ and the coefficient of t^d is $W(D)$.

A proof of Lemma 4.3 follows from Formula (29) of [BN1] and Lemma 4.1.

5 Products of weight systems

For a chord diagram D define

$$W_{r, \underline{x}}(D) := \sum_{D=D_1 \cup \dots \cup D_r} x_1^{\deg D_1} \dots x_r^{\deg D_r} W(D_1 \dots D_r) \in R[x_1, \dots, x_r]. \quad (20)$$

It will follow from Proposition 5.1 and is easy to see directly that the map $W_{r, \underline{x}}$ is well defined. In each finite degree we want to obtain $W_{r, \underline{x}}$ as a linear combination of weight systems coming from the Lie algebras $\mathfrak{gl}_n^{\oplus r}$ and $\mathfrak{so}_n^{\oplus r}$. In the sequel we will suppress some indices and hope that this does not lead to confusion.

Let \mathfrak{g}_n be one of the Lie algebras \mathfrak{gl}_n or \mathfrak{so}_n , let τ be the defining representation and ω be the Casimir element from Lemma 4.2 or Lemma 4.3. Let $a_1, \dots, a_r \in R$, define $\Omega := (a_1\omega, \dots, a_r\omega) \in (\mathfrak{g}_n \otimes \mathfrak{g}_n)^{\oplus r} \subset (\mathfrak{g}_n^{\oplus r}) \otimes (\mathfrak{g}_n^{\oplus r})$ and let the $\mathfrak{g}_n^{\oplus r}$ -representation $\tau^{\otimes r}$ be the r -fold tensor power of the standard representation τ .

Proposition 5.1 *For $\mathfrak{g} = \mathfrak{gl}$ or \mathfrak{so} and $r \geq 1$ there exists a morphism of R -algebras*

$$W_{\mathfrak{g}, r} : A \longrightarrow R[x_1, \dots, x_r]^{S_r}[t]$$

such that for all choices of $a_1, \dots, a_r \in R, n \geq 2$ we have

$$W_{\mathfrak{g}_n^{\oplus r}, \Omega, \tau^{\otimes r}}(D) = n^r W_{\mathfrak{g}, r}(D)(a_1, \dots, a_r, n).$$

If D has degree d , then the degree in the indeterminate t of $W_{\mathfrak{g}, r}(D)$ is $\leq d$ and the coefficient of t^d is $W_{r, \underline{x}}(D)$.³

Proof: Formula (10) and Exercise 6.33 of [BN1] give us the formula

$$W_{\mathfrak{g}_n^{\oplus r}, \Omega, \tau^{\otimes r}}(D) = \sum_{D=D_1 \cup \dots \cup D_r} a_1^{\deg D_1} W_{\mathfrak{g}_n, \omega, \tau}(D_1) \dots a_r^{\deg D_r} W_{\mathfrak{g}_n, \omega, \tau}(D_r).$$

This formula is valid for all choices of $a_1, \dots, a_r \in R$ and $n \geq 2$ and $W_{\mathfrak{g}_n, \omega, \tau}(D_i)$ is a polynomial in n with vanishing constant term (see Lemma 4.2 and Lemma 4.3), so there exists a unique weight system $W_{\mathfrak{g}, r}$ with the defining property from the proposition. We again use the Lemmas 4.2 and 4.3 to see that $W_{r, \underline{x}}(D)$ is the coefficient of t^d in $W_{\mathfrak{g}, r}(D)$ ($\deg D = d$) and to see that $W_{\mathfrak{g}, r}$ is a morphism of algebras. We have $W_{\mathfrak{g}, r}(a) \in R[x_1, \dots, x_r]^{S_r}[t]$ for all $a \in A$ because A is cocommutative. \square

Recall that the algebra $R[x_1, \dots, x_r]^{S_r}$ is a polynomial algebra in the polynomials

$$G_{i, r} := x_1^i + \dots + x_r^i \quad (i = 1, \dots, r). \quad (21)$$

In order to be as explicit as possible in the proof of Theorem 3.1 we formulate the next lemma.

Lemma 5.1 *The coefficient functions of monomials in $G_{1, r}, \dots, G_{r, r}, t$ from $W_{\mathfrak{g}, r} : A_d \longrightarrow R[G_{1, r}, \dots, G_{r, r}]_d[t]$ are linear combinations of weight systems associated to the Lie algebras $\mathfrak{g}_n^{\oplus r}$.*

Proof: We give a short constructive proof and we do not care about the number of terms in the linear combination:

Let P_i ($i = 2, \dots, r + d + 2$) be the unique polynomial of degree $r + d$, such that $P_i(j) = \delta_{ij}$ ($j = 2, \dots, r + d + 2$). In order to expand the result in the desired basis we choose some R -linear projection $\pi : R[x_1, \dots, x_r] \longrightarrow R[G_{1, r}, \dots, G_{r, r}]_d$. Then for chord diagrams D of degree d we have

$$W_{\mathfrak{g}, r}(D) = \sum_{a_1, \dots, a_r, n=2}^{r+d+2} W_{\mathfrak{g}_n^{\oplus r}, (a_1 \omega, \dots, a_r \omega), \tau^{\otimes r}}(D) \pi(P_{a_1}(x_1) \dots P_{a_r}(x_r)) P_n(t) / n^r$$

and the coefficients of monomials in $G_{1, r}, \dots, G_{r, r}, t$ can be read from the right hand side as linear combinations of weight systems coming from the Lie algebras $\mathfrak{g}_n^{\oplus r}$. \square

³The constant term of $W_{\mathfrak{gl}, r}$ can be expressed in terms of immanent weight systems (see Section 6 of [BNG] and Section 5 of [KSA]).

6 Chromatic weight systems and $W_{r,\underline{x}}$

Let $f_i : \mathbb{N} \longrightarrow R$ ($i = 1, \dots, r$) be functions. Define $\eta_i : A \longrightarrow A$ by $\eta_i(a) := f_i(\deg a)a$. We will be interested in the weight system

$$W_{r,\underline{f}} := W^{\otimes r} \circ (\eta_1 \otimes \dots \otimes \eta_r) \circ \Delta_r. \quad (22)$$

Let $a_i \in R$ ($i = 1, \dots, r$). For the special choice $f_i(d) = a_i^d$ we have $W_{r,\underline{f}}(D) = W_{r,\underline{x}}(D)(a_1, \dots, a_r)$ because of Formula (20). It will soon turn out that other choices of the maps f_i will not lead to greater generality. Let us now introduce some terminology to describe the combinatorics underlying Formula (22).

Definition 6.1 *Let G be a graph with vertex set $M_0(G)$. We call a function $c : M_0(G) \longrightarrow \{1, \dots, r\}$ a vertex coloring of G with r colors if for each edge $]v_1; v_2[$ we have $c(v_1) \neq c(v_2)$. Define*

$$C_r(G) := \begin{cases} \{c : M_0(G) \longrightarrow \{1, \dots, r\} \mid c \text{ is a vertex coloring}\} & \text{if } G \neq \emptyset, \\ \{\text{id} : \{0\} \longrightarrow \{0\}\} & \text{if } G = \emptyset. \end{cases}$$

Notice that $W(D) = \#C_1(\Gamma(D))$. The next proposition will be responsible for the connection between $W_{r,\underline{f}}$ and chromatic weight systems.

Proposition 6.1 *The following formula holds for all chord diagrams D :*

$$W_{r,\underline{f}}(D) = \sum_{c \in C_r(\Gamma(D))} \prod_{i=1}^r f_i(\#c^{-1}(i)).$$

Proof: By definition we have for a chord diagram D :

$$W_{r,\underline{f}}(D) = \sum_{c : M_0(\Gamma(D)) \longrightarrow \{1, \dots, r\}} \prod_{i=1}^r f_i(\#c^{-1}(i)) W(D_{c,i}).$$

where $D_{c,i}$ denotes the subdiagram of D with $\{a \mid c(a) = i\}$ as the set of chords. But

$$\prod_{i=1}^r W(D_{c,i}) = \begin{cases} 1 & \text{if } c \text{ is a vertex coloring of } \Gamma(D), \\ 0 & \text{otherwise.} \end{cases}$$

□

Let G be a weighted graph with weight function w and define

$$\overline{W}_{r,\underline{f}}(G) := \sum_{c \in C_r(G)} \prod_{i=1}^r f_i \left(\sum_{v \in M_0(G)} \delta_{i,c(v)} w(v) \right). \quad (23)$$

Lemma 6.1 *Formula (23) defines a linear map $\overline{W}_{r,\underline{f}} : \mathcal{C} \longrightarrow R$.*

Proof: Let $e =]v_1; v_2[$ be an edge of G and recall the definition of G'_e and G''_e from Section 3. We can identify the sets

$$\{c \in C_r(G'_e) \mid c(v_1) \neq c(v_2)\} \text{ and } \{c \in C_r(G'_e) \mid c(v_1) = c(v_2)\}$$

with $C_r(G)$ and $C_r(G''_e)$ respectively. Let $c \in C_r(G'_e)$ with $c(v_1) = c(v_2)$ and denote by \bar{c} the corresponding element of $C_r(G''_e)$. Denote the weight functions of G'_e and G''_e by w and \bar{w} respectively. Then we have for $i = 1, \dots, r$:

$$\sum_{v \in M_0(G'_e)} \delta_{i,c(v)} w(v) = \sum_{v \in M_0(G''_e)} \delta_{i,\bar{c}(v)} \bar{w}(v),$$

because in the definition of G''_e the new contracted vertex has the weight $w(v_1) + w(v_2)$. A similar statement holds for $c \in C_r(G'_e)$ with $c(v_1) \neq c(v_2)$ and the corresponding element $\bar{c} \in C_r(G)$. This implies

$$\overline{W}_{r,\underline{f}}(G) = \overline{W}_{r,\underline{f}}(G'_e) - \overline{W}_{r,\underline{f}}(G''_e)$$

and completes the proof. \square

We have $\overline{W}_{r,\underline{f}}(\overline{\Gamma}(D)) = W_{r,\underline{f}}(D)$. Making the choice $f_i(d) = x_i^d$ we get a map $\overline{W}_{r,\underline{x}} : \mathcal{C} \longrightarrow R[x_1, \dots, x_r]^{S_r}$ and a commutative diagram with morphisms of graded algebras:

$$\begin{array}{ccc} A & \xrightarrow{\overline{\Gamma}} & C \\ W_{r,\underline{x}} \searrow & & \swarrow \overline{W}_{r,\underline{x}} \\ & R[x_1, \dots, x_r]^{S_r} & \end{array} \quad (24)$$

Now we prove that the weight systems $W_{r,\underline{x}}$ for all natural numbers r are as strong as the universal chromatic weight system.

Lemma 6.2 *The map $\overline{W}_{r,\underline{x}} : \mathcal{C}_{\leq r} \longrightarrow R[x_1, \dots, x_r]_{\leq r}^{S_r}$ is an isomorphism of vector spaces.*

Proof: By the definition of $\overline{W}_{r,\underline{x}}$ we have for single vertices of weight d :

$$\overline{W}_{r,\underline{x}} \left(\begin{smallmatrix} d \\ \bullet \end{smallmatrix} \right) = x_1^d + \dots + x_r^d = G_{d,r}. \quad (25)$$

This formula shows that the graded map $\overline{W}_{r,\underline{x}}$ is surjective and we have

$$\dim \mathcal{C}_{\leq r} \leq \dim R[x_1, \dots, x_r]_{\leq r}^{S_r}$$

because $\deg G_{d,r} = d$ and $\mathcal{C}_{\leq r}$ is spanned by graphs of weight $\leq r$ consisting of isolated vertices. \square

Proof of Theorem 3.1 By Proposition 5.1 and Lemma 5.1 the weight system $W_{d,\underline{x}}$ is a linear combination of the form described in Theorem 3.1. By the commutative triangle (24) and Lemma 6.2 we see that every weight system of degree d that factors through \mathcal{C}_d also factors through $R[x_1, \dots, x_d]_d^{S_d}$. This proves Theorem 3.1. \square

As mentioned before the weight system $W_{r,\underline{f}}$ is not a far-reaching generalization of $W_{r,\underline{x}}$.

Remark 6.1 *The weight system $W_{r,\underline{f}}$ is also a chromatic weight system; so for degree d , it factors through $R[x_1, \dots, x_d]_d^{S_d}$.*

The restriction to A_d of the coefficients of monomials in x_1, \dots, x_r in $W_{r,\underline{x}}$ are weight systems lying in the algebra generated by the weight systems of finite degree $W|A_i$ ($i = 1, 2, \dots$). Now Lemma 6.2 implies immediately the following proposition that can also be proved more directly.

Proposition 6.2 *There exists an isomorphism of graded algebras*

$$R[x_1, x_2, \dots] \longrightarrow \bigoplus_{i=0}^{\infty} \mathcal{C}_i^* \quad (\deg x_i = i)$$

such that the composition with the inclusion of algebras

$$\bigoplus_{i=0}^{\infty} (\overline{\Gamma}|A_i)^* : \bigoplus_{i=0}^{\infty} \mathcal{C}_i^* \longrightarrow \bigoplus_{i=0}^{\infty} A_i^*$$

is given by $x_i \mapsto (W|A_i) \in A_i^$.*

The system of generators for the algebra spanned by chromatic weight systems of finite degree we used in the proposition consists for $d \geq 2$ not of primitive elements of $\bigoplus_i A_i^*$.

7 Embeddings into a 2-sphere

By Section 6.3 of [BN1] there is a connection between the weight systems $W_{\mathfrak{gl}}$ and $W_{\mathfrak{so}}$ and surfaces. The description of $W_{\mathfrak{gl}}$ and $W_{\mathfrak{so}}$ for trivalent graphs instead of chord diagrams allows easily also to consider the adjoint representation instead of the defining representation. We will use [BN2] to give a formula for the value of the universal chromatic weight system on certain trivalent diagrams in terms of embeddings into a 2-sphere.

A special case of Proposition 6.1 is the following:

Remark 7.1 *Denote the map $W_{r,f}$ for the choice $r = 2$ and $f_1 = f_2 = 1$ by $W_{2,1}$. Then the formula $W_{2,1}(D) = \#C_2(\bar{\Gamma}(D))$ for chord diagrams D from Proposition 6.1 can be restated as*

$$W_{2,1}(D) = \begin{cases} 0 & \text{if } \Gamma(D) \text{ is not bipartite,} \\ 2^{\#\pi_0(\Gamma(D))} & \text{otherwise.} \end{cases}$$

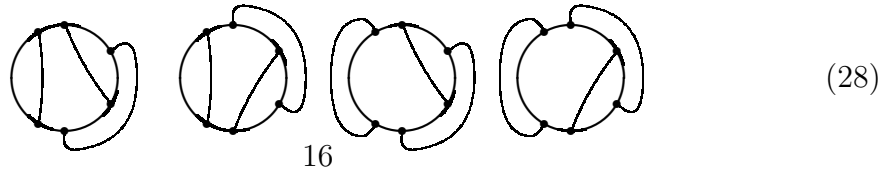
Let D be a trivalent diagram. We choose a cyclic order for the trivalent vertices on S^1 such that it is counterclockwise in our pictures. If there exists an embedding of D into an oriented 2-sphere, then let $\sigma(D)$ be the number of trivalent vertices of D for which the cyclic order does not coincide with the counterclockwise cyclic order induced by the chosen embedding. Let $\sigma(D)$ be arbitrary if no embedding exists. Then by [BN2] the formula

$$E(D) := (-1)^{\sigma(D)} \# \{\text{embeddings of } D \text{ into an oriented 2-sphere}\} \quad (26)$$

defines a weight system. By Remark 7.1 we have

$$W_{2,1}(D) = E(D) \quad (27)$$

for all chord diagrams D hence for all elements of A . As an example the embeddings of a chord diagram D with $E(D) = 4$ are shown in Picture (28).



Now we can state the 'geometric version' of Theorem 3.1.

Proposition 7.1 *For $a \in P(A)_d$ we have*

$$\bar{\Gamma}(a) = \frac{1}{2}E(a) \mathring{d}.$$

where \mathring{d} denotes a single vertex of weight d .

Proof: By definition of $W_{r,\underline{f}}$ we have for $a \in P(A)_d$:

$$W_{r,\underline{f}}(a) = \sum_{i=1}^r f_i(d) \prod_{\substack{j=1 \\ j \neq i}}^r f_j(0) W(a).$$

The previous formula implies $W_{2,1}(a) = 2W(a)$ and by Formula (27) we have

$$W_{r,\underline{f}}(a) = \frac{1}{2} \sum_{i=1}^r f_i(d) \prod_{\substack{j=1 \\ j \neq i}}^r f_j(0) E(a).$$

In particular, $W_{r,\underline{x}}(a) = \frac{1}{2}(x_1^d + \dots + x_r^d)E(a)$. This implies by the commutative triangle (24), Lemma 6.2 and Formula (25) the desired formula for $\bar{\Gamma}(a)$. \square

Notice that Proposition 7.1 gives us a formula for $\bar{\Gamma}$ for all elements of $\bigcup_{i=0}^{\infty} \mathcal{M}^i$ (see Formula (6)) in terms of E . As an example consider the set

$$\mathcal{T} := \left\{ \begin{array}{l} \text{planar trivalent diagrams } D \\ \text{such that } D \setminus S^1 \text{ is a tree} \end{array} \right\}. \quad (29)$$

We have $\mathcal{T} \subset \mathcal{M}$ and elements of \mathcal{T} have exactly two embeddings into a 2-sphere. By Proposition 7.1 we have

$$\bar{\Gamma}(D) = \mathring{d}. \quad (30)$$

for elements $D \in \mathcal{T}$ of degree d . The element of \mathcal{T} shown in Picture (31) is for degree d equal to the element $(-1)^d p_d$ of Proposition 2 of [CDL3].



(31)

This leads us to the following remark.

Remark 7.2 *The subalgebra of A generated by \mathcal{T} is also generated by the chord diagrams D such that $\Gamma(D)$ is a tree. It is called the forest algebra in [CDL3].*

Remark 7.2 follows from [Vo2], Application on page 7 and [CDL3], Theorem 1 and Proposition 2.

8 Exceptional Lie superalgebras

In [Vo2] the independence of the weight systems associated to the Lie superalgebras of Part III of Table (32) (Theorem 7.1 of [Vo2]) and the insufficiency of semisimple Lie superalgebras with Casimir to generate all weight systems (Theorem 7.4 of [Vo2]) are proved.

$$III \left\{ \begin{array}{l} \mathfrak{sl}(m, n), \quad (m \neq n) \\ \mathfrak{osp}(m, n), \quad (n \text{ even}) \\ \text{exceptional Lie algebras } E_6, E_7, E_8, F_4, G_2 \\ D(2, 1, \alpha), \quad (\alpha \in R \setminus \{0, -1\}) \\ \text{exceptional Lie superalgebras } G(3), F(4) \end{array} \right\} \begin{array}{l} I \\ II \end{array} \quad (32)$$

In Theorem 3.1 we discovered a relation between chromatic weight systems and the Lie superalgebras of Part I of Table (32) ⁴. In this section we consider Part II of the list. Using the methods from [Vo2] and Proposition 7.1 we can prove the following:

Proposition 8.1 *Assume that the ground field R is algebraically closed and of characteristic 0. For all $d \geq 8$ there exists an element $P_d \in P(A)_d$ with $\overline{\Gamma}(P_d) \neq 0$ and the following property: Let \mathfrak{g} be a simple Lie superalgebra from Part II of Table (32) and let ω be a Casimir element for \mathfrak{g} . Then for all finite-dimensional representations ρ of \mathfrak{g} we have $W_{\mathfrak{g}, \omega, \rho}(P_d) = 0$.*

⁴The weight systems associated to the standard representation of a Lie superalgebra from Part I of Table (32) only depends on the superdimension $m - n$ of the standard representation, so they are already determined by \mathfrak{sl}_n and \mathfrak{so}_n . Using this it can easily be shown that an equivalent formulation of Theorem 3.1 could use each of the four classical series of simple Lie algebras.

Proof: In [Vo2] a commutative graded R -algebra Λ and the structure of a graded Λ -module on $P(A)_{\geq 2}$ are defined. Let \mathfrak{g} be an exceptional Lie algebra or a Lie superalgebra of type $G(3)$ or $F(4)$ and let $\omega = \sum_i x_i \otimes y_i$ be the Casimir element of \mathfrak{g} associated to the Killing form. Then the endomorphism $\sum_i \text{ad}_{\mathfrak{g}}(x_i) \circ \text{ad}_{\mathfrak{g}}(y_i)$ is equal to $\text{id}_{\mathfrak{g}}$. Theorem 6.1 of [Vo2] and the subsequent remarks say that there exists a unique homomorphism of graded algebras $\chi_{\mathfrak{g}} : \Lambda \longrightarrow R[t]$ such that for all $\alpha \in \Lambda$, for all \mathfrak{g} -representations ρ and for all $a \in P(A)_{\geq 2}$ we have

$$W_{\mathfrak{g},\omega,\rho}(\alpha a) = \chi_{\mathfrak{g}}(\alpha)(1/2)W_{\mathfrak{g},\omega,\rho}(a).$$

Certain elements x_1, x_3, x_5, \dots of Λ generate a subalgebra Λ_0 , on which formulas for the maps $\chi_{\mathfrak{g}}$ are known. When we define $t := x_1/2 \in \Lambda_0$ we always have $\chi_{\mathfrak{g}}(t) = t \in R[t]$. A formula for $\chi_{\mathfrak{g}}(x_n)$ for exceptional Lie algebras is given by Theorem 6.9 of [Vo2]. Following Section 6.8 and the remark on page 40 of [Vo2] we determine for all $n > 0$:

$$\chi_{G(3)}(x_n) = \frac{1}{6} (3 + 2^{n+1} + 5(-1)^{n+1}) t^n$$

and

$$\chi_{F(4)}(x_n) = \frac{1}{16} (8 + 2^{n+1} + 8 + 27(-2/3)^{n+1}) t^n.$$

In particular, we have

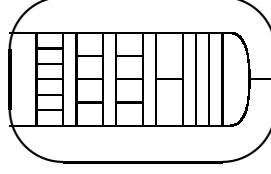
$$\chi_{G(3)}(x_3 - 4t^3) = 0 \quad \text{and} \quad \chi_{F(4)}(3x_3 - 7t^3) = 0.$$

Let p_2 be the trivalent diagram of degree 2 shown in Picture (31). As in the proof of Theorems 7.1 and 7.4 of [Vo2] we see that for all $d \geq 23$ the element

$$\begin{aligned} \overline{P}_d &= t^{d-23}(8x_3 - 7t^3)(81x_3 - 64t^3)(15x_3 - 11t^3)(81x_3 - 85t^3) \\ &\quad \times (24x_3 - 41t^3)(x_3 - 4t^3)(3x_3 - 7t^3) \cdot p_2 \end{aligned}$$

has the property $W_{\mathfrak{g},\omega,\rho}(\overline{P}_d) = 0$ for all simple Lie superalgebras we consider in the proposition. The argument concerning the Lie superalgebras of type $D(2, 1, \alpha)$ is contained in the proof of Theorem 7.4 of [Vo2].

We can prove that for all monomials x in t, x_3, x_5, \dots the trivalent diagram representing $x \cdot p_2$ has exactly two embeddings into a 2-sphere. As examples, for all degrees $d \geq 2$ the element of Picture (31) is equal to $t^{d-2}p_2$ and $t^3x_3x_5^2x_7 \cdot p_2$ is shown in Picture (33).



(33)

This implies by Proposition 7.1:

$$\bar{\Gamma}(\bar{P}_d) = (8-7)(81-64)(15-11)(81-85)(24-41)(1-4)(3-7)\dot{d} \neq 0.$$

and completes the proof for all $d \geq 23$. The better bound $d \geq 8$ is obtained by a slightly longer computation using the following sequence of elements:

$$P_d := t^{d-8} \cdot (32t^6 - 71t^3x_3 + 18x_3^2 - 45tx_5) \cdot p_2, \quad (d \geq 8).$$

□

With some technical arguments stated in the Appendix Proposition 8.1 can be generalized to direct sums of Lie superalgebras over fields of characteristic 0. This will prove the following complementary result to Theorem 3.1.

Theorem 8.1 *For all degrees $d \geq 8$ the map $\bar{\Gamma}|P(A)_d$ is not a linear combination of weight systems coming from finite direct sums of Lie superalgebras from Part II of Table (32).*

9 The bialgebra of framed knots

Let $R[\mathbb{Z}/(2)]$ be the group algebra of the group with two elements considered as a graded bialgebra over R with all elements having degree 0. Define

$$\hat{A} := A \otimes R[\mathbb{Z}/(2)]. \quad (34)$$

We can think of elements from \hat{A} as linear combinations of chord diagrams, each chord diagram being colored with an element of $\mathbb{Z}/(2)$. We call the element of $\mathbb{Z}/(2)$ the residue of the diagram.

Let us explain the origin of the graded bialgebra structure on \hat{A} . By framed oriented knots we will always mean tame framed oriented knots considered up to orientation preserving diffeomorphisms of \mathbb{R}^3 . Denote the vector space freely generated by framed oriented knots by \mathcal{K} . The set of framed knots is a semi-group with the connected sum of framed knots as multiplication. This turns \mathcal{K} into a bialgebra for which framed knots are group-like elements.

Singular framed knots will be considered here as a short notation for certain linear combinations of framed knots given by the 'desingularisation rule' from Picture (35). This rule is valid for three singular framed knots that differ only inside a small ball like shown in Picture (35). The framing of the shown parts points to the reader.

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \quad (35)$$

Let \mathcal{K}_i be the subspace of \mathcal{K} spanned by all singular framed knots with i double points. This defines a decreasing sequence of vector spaces:

$$\mathcal{K} = \mathcal{K}_0 \supseteq \mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \dots \quad (36)$$

A translation of Proposition 4 from [Vo1] to framed knots tells us that this sequence is a filtration of the bialgebra \mathcal{K} . The connected sum of singular framed knots is expressed in the same way as for framed knots without double points and the coproduct of a singular framed knot can be expressed like in Formula (3) of [Vo1]. Denote the graded bialgebra associated to \mathcal{K} by

$$\text{gr } \mathcal{K} := \bigoplus_{i=0}^{\infty} \mathcal{K}_i / \mathcal{K}_{i+1}. \quad (37)$$

Recall the construction of a singular framed knot K_D as a 'realization' of a chord diagram D with residue from [KaT]. We summarize it in Picture (38).

$$\text{Replace } \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \text{ by } \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \text{ and } \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \text{ by } \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \quad (38)$$

If $\deg D = d$, then $K_D \in \mathcal{K}_d/\mathcal{K}_{d+1}$ is well defined. This gives us an isomorphism of graded bialgebras $\psi : \hat{A} \longrightarrow \text{gr } \mathcal{K}$. The special case $\dim \mathcal{K}_0/\mathcal{K}_1 = 2$ may be proved directly. In general the proof that ψ is injective is based on Part (1) of the next theorem. In order to formulate it let us introduce some notation. We can pass from a graded vector space $B = \bigoplus_{i=0}^{\infty} B_i$ to its completion $B^c := \prod_{i=0}^{\infty} B_i$. We will use the notation $\sum_{i=0}^{\infty} b_i h^i$ with $b_i \in B_i$ and a formal parameter h for elements of B^c . If B is a graded algebra, then B^c is an algebra. The elements of B^c are multiplied like formal power series.

Theorem 9.1 (Kontsevich) (1) *There exists a linear map $Z : \mathcal{K} \longrightarrow \hat{A}^c$ such that for any realisation K_D of a chord diagram D with residue we have*

$$Z(K_D) = Dh^{\deg D} + \text{terms of higher degree.}$$

(2) *With the comultiplication defined by $\Delta(\sum_{i=0}^{\infty} a_i h^i) := \sum_{i=0}^{\infty} \Delta(a_i) h^i$ the image of the map Z is a bialgebra and $Z : \mathcal{K} \longrightarrow \text{image } Z$ is a morphism of bialgebras.*

For a proof of Part (1) of Theorem 9.1 see [KaT] and also [BN1] and [LM3]. For a proof of Part (2) and a proof that Z is already defined over the rational numbers one may use the explicit description of Z from [KaT] and Theorem A'' from [Dri]. We have used a normalization of Z different from the one used in [KaT] in order to turn Z into a morphism of algebras⁵. Our normalization specializes to the one in [BN1] when we forget the residues and introduce the 'framing independence' relation (see [LM3], Theorem 6 and Theorem 7). For reasons that will become clear in the next section the map Z is called the *universal Vassiliev invariant*. It is an open question whether Z is injective.

10 Vassiliev invariants

Now we take a look at the dual notions of the filtered bialgebra of framed knots and the graded bialgebra of chord diagrams and translate the results of the previous section.

Definition 10.1 *We call a linear map $v : \mathcal{K} \longrightarrow R$ a Vassiliev invariant of degree d if $v(\mathcal{K}_{d+1}) = \{0\}$. Let \mathcal{V}_d be the space of Vassiliev invariants of degree d and $\mathcal{V} := \bigcup_{i=0}^{\infty} \mathcal{V}_d$ the space of all Vassiliev invariants.*

⁵We give more details in the proof of Proposition 12.1.

The vector spaces \mathcal{V}_i form an increasing sequence

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \dots \subseteq \mathcal{V}. \quad (39)$$

The product of $v_1 \in \mathcal{V}_k$ and $v_2 \in \mathcal{V}_l$ is defined to be pointwise on framed knots K without double points: $(v_1 v_2)(K) := v_1(K) v_2(K)$. With this definition \mathcal{V} becomes a subalgebra of \mathcal{K}^* . Now $v_1 v_2$ is in \mathcal{V}_{k+l} because the coproduct of $\text{gr } \mathcal{K}$ respects the grading. Define $\mathcal{V}_{-1} := \{0\}$. Theorem 9.1 enables us to choose nice representatives for elements of $\mathcal{V}_d / \mathcal{V}_{d-1}$ ($d \geq 0$). This will allow us to turn \mathcal{V} into a graded algebra as follows: Denote the projection $\hat{A}^c \longrightarrow \hat{A}_d$ by p_d . Given $w \in \hat{A}_d^*$, we can define $\hat{\Phi}_d(w) \in \mathcal{V}_d$ by

$$\hat{\Phi}_d(w)(K) = (w \circ p_d \circ Z)(K). \quad (40)$$

The image $\mathcal{V}_d^{\text{can}} := \hat{\Phi}_d(\hat{A}_d^*)$ of the linear map $\hat{\Phi}_d$ is called the space of canonical Vassiliev invariants of degree d . We state the translation of Theorem 9.1 to this setting (see [BN1], Theorem 9).

Proposition 10.1 (1) *The map*

$$\bigoplus_{i=0}^{\infty} \hat{\Phi}_i : \bigoplus_{i=0}^{\infty} \hat{A}_i^* \longrightarrow \bigoplus_{i=0}^{\infty} \mathcal{V}_i^{\text{can}} = \mathcal{V}$$

is an isomorphism of algebras.

(2) *The image of the space of primitive elements $P(\bigoplus_{i=0}^{\infty} \hat{A}_i^*)$ is the space of Vassiliev invariants that are additive with respect to the connected sum of framed oriented knots.*

Now we pass to completions.

Definition 10.2 *A formal sum $v = \sum_{i=0}^{\infty} v_i h^i$ with $v_i \in \mathcal{V}_i$ is called a Vassiliev series. The Vassiliev series v is called canonical if $v \in \prod_{i=0}^{\infty} \mathcal{V}_i^{\text{can}}$. The Vassiliev series v is called multiplicative if the map $v : \mathcal{K} \longrightarrow R[[h]]$ is a morphism of algebras.*

We have a projection $\rho_d : \hat{A}_d \longrightarrow A_d$ defined by forgetting residues. The dual map allows to lift elements from A_d^* to \hat{A}_d^* .

Definition 10.3 (1) *For a weight system w define a canonical Vassiliev series by*

$$\Phi(w) := \sum_{i=0}^{\infty} (w \circ \rho_i \circ p_i \circ Z) h^i$$

(2) For a Vassiliev series $v = \sum_{i=0}^{\infty} v_i h^i$ define a weight system $\Psi(v) \in A^*$ by $\Psi(v)(D) := v_d(K_D)$ where D is a chord diagram of degree d and K_D is a realization of D with residue $\overline{0}$.

By Part (1) of Theorem 9.1 we have for every weight system w :

$$\Psi(\Phi(w)) = w. \quad (41)$$

A consequence of Part (2) of Theorem 9.1 is the following proposition (see also [BNG], Proposition 2.9).

Proposition 10.2 (1) *The canonical Vassiliev series $\Phi(w)$ is multiplicative if and only if w is multiplicative.*

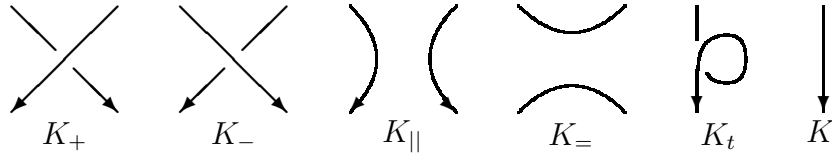
(2) *For weight systems w_1 and w_2 we have $\Phi(w_1 w_2) = \Phi(w_1) \Phi(w_2)$.*

11 The knot polynomial Υ

Define a linear map $\Upsilon : \mathcal{K} \longrightarrow R[y, y^{-1}]$ by:

$$\begin{aligned} (1) \quad & \Upsilon(K_+) - \Upsilon(K_-) = (y - y^{-1})\Upsilon(K_{||1})\Upsilon(K_{||2}), \\ (2) \quad & \Upsilon(K_t) = y\Upsilon(K_{|}), \\ (3) \quad & \Upsilon(O) = 1 \end{aligned}$$

where we use two knots K_+ and K_- and a link $K_{||}$ with two connected components that differ only in a ball like shown in Picture (42) for the first equation. We denote the two knots that are the connected components of $K_{||}$ by $K_{||1}$ and $K_{||2}$. The symbols K_t and $K_{|}$ from the second equation are also explained by Picture (42). The symbol O from the third equation denotes the oriented unknot with 0-framing. Proposition 12.2 says that Υ is well defined.

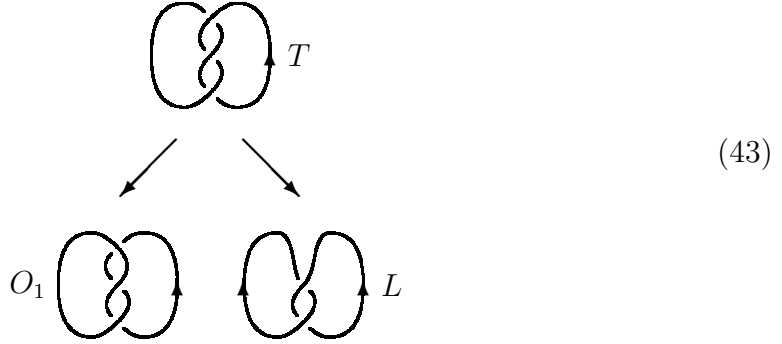


$$\quad (42)$$

Define $\Upsilon_h(K) := \Upsilon(K)(\exp(h/2))$ (this shall mean that $y^{\pm 1}$ is substituted by $\exp(\pm h/2)$). Then Υ_h is a power series and the coefficients of powers of h are knot invariants with values in R . Now we formulate our second main result.

Theorem 11.1 *The algebra of all chromatic weight systems of finite degree is mapped by Φ to the subalgebra of \mathcal{V} generated by the coefficients of Υ_h . In particular, the Vassiliev invariants corresponding to all chromatic weight systems distinguish exactly the same knots as the polynomial Υ .*

Theorem 11.1 will follow from Proposition 6.2, Part (2) of Proposition 10.2 and Proposition 12.2. Let us now consider an example.



Using the tree in Picture (43) we can determine the value of Υ on the shown trefoil knot T as follows. By Relations (2) and (3) we have $\Upsilon(O_1) = y$. The two connected components of the link L from Picture (43) are unknots O with 0-framing, so we use Relations (1) and (3) to compute

$$\Upsilon(T) = \Upsilon(O_1) + (y - y^{-1})\Upsilon(O)\Upsilon(O) = 2y - y^{-1}. \quad (44)$$

The following remark is easy to prove.

Remark 11.1 *Let K be an oriented framed knot with mirror image \overline{K} . Then we have*

$$\Upsilon(K)(y) = \Upsilon(\overline{K})(y^{-1}) \quad (45)$$

From the remark and Formula (44) we see that the polynomial Υ can distinguish at least one framed knot from its mirror image. ⁶

⁶ The knot invariant Υ has also been studied in Section 4 of [Kne] in a framing independent normalization. Corollary 2 of [Kne] implies that the set of all polynomials that appear as values $\Upsilon(K)$ for some oriented framed knot K is equal to $\bigcup_{i \in \mathbb{Z}} y^i (1 + (y^2 - 1)\mathbb{Z}[y^2, y^{-2}])$.

12 Canonical Vassiliev series coming from \mathfrak{gl}_n

Now we translate our results about weight systems into results about the corresponding canonical Vassiliev series. The next proposition is proved in [LM1] for links without framing with analytical methods. We use the same ideas to give a proof for framed links with the help of the algebraic description of the universal Vassiliev invariant Z . Recall the notation from Picture (42).

Proposition 12.1 *a) For all $n \geq 1$ there exists an invariant H of framed oriented links with values in $R[[h]]$ satisfying the following three relations:*

- (1) $H(K_+) - H(K_-) = (\exp(h/2) - \exp(-h/2)) H(K_{||}),$
- (2) $H(K_t) = \exp(nh/2) H(K_{|}),$
- (3) $H(O) = 1.$

b) For framed oriented knots K we have

$$H(K) = \Phi(W_{\mathfrak{gl}_n, X, \tau})(K)/n.$$

c) Define $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$ for invertible elements $q \in R[[h]]^$. Then we have for the disjoint union $K_1 \amalg K_2$ of framed oriented links K_1 and K_2 :*

$$H(K_1 \amalg K_2) = [n]_{\exp(h/2)} H(K_1) H(K_2).$$

Proof: We start with a remark concerning generalisations of Theorem 9.1 and Lemma 2.1 of this article. As a general reference we use [KaT] and also Chapters XI and XX of [Kas].

The objects of all categories that we consider in this proof are finite sequences of '+' and '-' symbols. There exists a category $\hat{A}_t[[h]]$ in which the morphisms are represented by formal power series of chord diagrams on circles and intervals with residue (see Section 2 of [KaT] for a precise definition of the diagrams we use here and for a formula for the residue of the composition of diagrams).

The universal Vassiliev invariant Z_t is a functor from the category \mathcal{T} of framed oriented tangles to $\hat{A}_t[[h]]$. In \mathcal{T} we have links as the endomorphisms of the empty sequence and for knots K we have

$$Z(K) = Z_t(K)/Z_t(O). \tag{46}$$

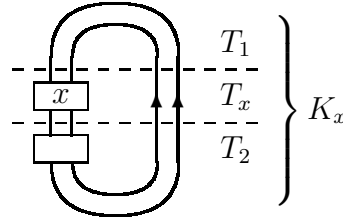
In Equation (46) the multiplication is induced by the connected sum of chord diagrams with residue and we use the inclusion of \hat{A}^c into the endomorphisms of the empty sequence in $\hat{A}_t[[h]]$.

The weight system $W_{\mathfrak{g},\omega,\rho}$ associated to a Lie superalgebra \mathfrak{g} with Casimir element ω and representation $\rho : \mathfrak{g} \longrightarrow \text{End } M$ can be generalized to a functor also called $W_{\mathfrak{g},\omega,\rho}$ from the category $\hat{A}_t[[h]]$ to a category $D_{\mathfrak{g}}(M, M^*)[[h]]$. In order to describe the morphisms in the last category we associate to an object $(\epsilon_1, \dots, \epsilon_k)$ ($\epsilon_i \in \{+, -\}$) of $D_{\mathfrak{g}}(M, M^*)[[h]]$ the module $M^{\epsilon_1} \otimes \dots \otimes M^{\epsilon_k}$ where M^+ denotes M and M^- denotes the dual module M^* . The 'empty tensor product' will be R . The morphisms between two objects of $D_{\mathfrak{g}}(M, M^*)[[h]]$ are formal power series of \mathfrak{g} -linear maps between the corresponding modules. In the sequel we assume $\text{End}_{\mathfrak{g}}(M) \cong R$. Then by Formula (46) and Proposition 2.1.6 of [LM1] we have for framed knots K :

$$\Phi(W_{\mathfrak{g},\omega,\rho})(K) = \frac{\dim M}{W_{\mathfrak{g},\omega,\rho}(Z_t(O))} W_{\mathfrak{g},\omega,\rho}(Z_t(K)). \quad (47)$$

We use the canonical isomorphism $R[[h]] \cong \text{End}_{D_{\mathfrak{g}}(M, M^*)[[h]]}(\emptyset)$ in Equation (47) where ' \emptyset ' denotes the empty sequence.

After this long preparation we start with a simple proof of the proposition. We divide each link K_x ($x \in \{+, -, ||\}$) from Relation (1) for H into three tangles T_1 , T_x and T_2 as shown in Picture (48). The tangles T_1 and T_2 are the same in all three cases and the part of the tangle T_x denoted by a box labeled x in Picture (48) looks like the corresponding part in Picture (42).



$$\left. \begin{array}{c} \text{---} T_1 \text{---} \\ \text{---} T_x \text{---} \\ \text{---} T_2 \text{---} \end{array} \right\} K_x \quad (48)$$

Now we make the special choice $\mathfrak{g} = \mathfrak{gl}_n$, $\omega = X$ and $\rho = \tau$ from Lemma 4.2. We define H for framed oriented links K by

$$H(K) := \frac{W_{\mathfrak{gl}_n, X, \tau}(Z_t(K))}{W_{\mathfrak{gl}_n, X, \tau}(Z_t(O))}. \quad (49)$$

Define \mathfrak{g} -linear maps $\varphi_i := W_{\mathfrak{gl}_n, X, \tau}(Z_t(T_i))$ and

$$\varphi_x := W_{\mathfrak{gl}_n, X, \tau}(Z_t(T_x)) \in \text{End}(M^{\otimes 2} \otimes M^{*\otimes 2})[[h]].$$

By the explicit description of the universal Vassiliev invariant Z_t (see Theorem 4.7 of [KaT]) we have

$$\begin{aligned}\varphi_+ &= (\exp(hX/2) \circ X) \otimes \text{id}_{M^*} \otimes \text{id}_{M^*}, \\ \varphi_- &= (\exp(-hX/2) \circ X) \otimes \text{id}_{M^*} \otimes \text{id}_{M^*}, \\ \varphi_{||} &= \text{id}_M \otimes \text{id}_M \otimes \text{id}_{M^*} \otimes \text{id}_{M^*}.\end{aligned}$$

In these formulas no associators appear because in Picture (48) the box labeled x is on the left side. A short computation in the algebra $(\text{End } M \otimes \text{End } M)[[h]]$ yields

$$(\exp(hX/2) - \exp(-hX/2)) \circ X = (\exp(h/2) - \exp(-h/2)) \text{id} \otimes \text{id}.$$

This implies Relation (1) because we have $(\varphi_1 \circ \varphi_x \circ \varphi_2)/W_{\mathfrak{gl}_n, X, \tau}(Z_t(O)) = H(K_x)$. Property (2) is proved by a similar computation and Property (3) follows from Equation (49).

By Equation (47) the invariant H is an extension of $\Phi(W_{\mathfrak{gl}_n, X, \tau})/n$ from knots to links. This proves Part b) of the theorem.

A computation for the disjoint union of two unknots using Relations (1)–(3) yields $H(O \amalg O) = [n]_{\exp(h/2)} H(O)^2$.

The functoriality of $W_{\mathfrak{gl}_n, X, \tau} \circ Z_t$ now allows to determine

$$W_{\mathfrak{gl}_n, X, \tau}(Z_t(O)) = [n]_{\exp(h/2)} \tag{50}$$

and gives us the general formula for the disjoint union of links for H ⁷. \square

This again proves that the Homfly polynomial is well defined. The next proposition is a consequence of a connection between H and Υ .

Proposition 12.2 *The invariant Υ is well-defined and for all framed oriented knots K we have*

$$\Phi(W)(K) = \Upsilon(K)(\exp(h/2)).$$

⁷By Theorem 7.2 of [KaT] or Theorem 10 of [LM3] we see that the factor $[n]_{\exp(h/2)}$ from Formula (50) appears also as a quantum dimension. For the more direct computations using R -matrices see [Tur].

Proof: By Lemma 4.2 we have for all n :

$$\Phi(W_{gl_n, X, \tau})/n = \sum_{i=0}^{\infty} \sum_{j=0}^i p_{ij} n^j h^i$$

with uniquely determined Vassiliev invariants p_{ij} of degree i and

$$\Phi(W) = \sum_{i=0}^{\infty} p_{ii} h^i. \quad (51)$$

From the Relations (1)-(3) for H from Lemma 12.1 we can derive relations for $\Phi(W)$ as follows:

Recall from Part I, Section 5 of [Kau] that we can calculate $H(K)$ for every link K with the help of a certain labeled binary tree with root, a so-called skein-tree. The vertices of such a tree are labeled with link diagrams, the root is labeled with a diagram for K , the leaves are labeled with diagrams of unknotted unlinked framed circles, and for the link diagrams labeling inner vertices there exists a crossing such that the relation shown in Picture (52) is satisfied. An example for a part of a skein-tree is shown in Picture (43).



Notice that $H(K_{||})$ is multiplied in Relation (1) with

$$\exp(h/2) - \exp(-h/2) = h + \text{terms of degree } > 1$$

and for the factor $[n]_{\exp(h/2)}$ for the disjoint union of links we have

$$\frac{\exp(nh/2) - \exp(-nh/2)}{\exp(h/2) - \exp(-h/2)} = n \sum_{\nu=0}^{\infty} \frac{(nh/2)^{2\nu}}{(2\nu+1)!} + \text{terms } a_{ij} n^i h^j \text{ with } i \leq j.$$

This implies that in a skein-tree for the calculation of $H(K)$ the subtree for the link $K_{||}$ can only give a contribution to the highest powers in n if the number of

components of $K_{||}$ is greater than the number of components of K_+ (and K_-). This is the case exactly if the two strands of the distinguished crossing of the link K_+ belong to the same connected component. Using this argument we see that $\Phi(W)$ satisfies the Relations (1)-(3) for Υ with the parameter $y = \exp(h/2)$. \square

This completes the proof of Theorem 11.1. From Part (1) of Proposition 10.2 and Formula (41) we get the following corollary to Proposition 12.2.

Corollary 12.1 (1) *The linear map Υ is a morphism of algebras.*
(2) *The power series Υ_h is a canonical Vassiliev series with corresponding weight system $\Psi(\Upsilon_h) = W$.*

A reason why we mainly considered knots and not links is given in the following remark that can be proved in the same way as Proposition 12.2.

Remark 12.1 *A generalization of Υ to links suggested by Formula (51) is not useful because we would have $\Upsilon(L) = \Upsilon(K_1) \cdot \dots \cdot \Upsilon(K_l)$ where K_1, \dots, K_l are the knots that are the connected components of the link L .*

Now consider the Lie algebra \mathfrak{so}_n with Casimir element ω and representation τ as in Lemma 4.3. Recall again the symbols from Picture (42).

Proposition 12.3 *a) For all $n \geq 2$ there exists an invariant F of framed oriented links with values in $R[[h]]$ satisfying the following three relations:*

- (1) $F(K_+) - F(K_-) = (\exp(h/2) - \exp(-h/2)) (F(K_{||}) - F(K_=))$,
- (2) $F(K_t) = \exp((n-1)h/2) F(K_{|})$,
- (3) $F(O) = 1$

where we may choose any orientation for $K_=$.

b) For framed oriented knots K we have

$$F(K) = \Phi(W_{\mathfrak{so}_n, \omega, \tau})(K)/n.$$

c) For the disjoint union of framed oriented links K_1 and K_2 we have

$$F(K_1 \amalg K_2) = ([n-1]_{\exp(h/2)} + 1) F(K_1) F(K_2).$$

A proof of Proposition 12.3 is only slightly more complicated than the proof of Proposition 12.1 (see Theorem 3.6 of [LM2]). The invariant F does not depend on the orientation of framed links. It is one of the standard parametrizations of the Kauffman polynomial (see [Kau]). Because of Lemma 4.3 we could also use it instead of H to prove Proposition 12.2. In particular, Υ is also a specialization of the Kauffman polynomial.

A Appendix

In order to generalize Proposition 8.1 from a simple Lie superalgebra to a direct sum $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$ of simple Lie superalgebras \mathfrak{g}_i we have to use a relation between the description of \mathfrak{g} -modules and the description of \mathfrak{g}_i -modules. A technical problem is that a finite-dimensional \mathfrak{g} -module need not be the direct sum of simple \mathfrak{g} -modules. To solve this problem recall the definition of a Grothendieck ring (see Part 16B of [CuR]). By $G_0(\mathfrak{g})$ we will denote the Grothendieck ring associated to the category of finite dimensional \mathfrak{g} -modules using all short exact sequences of \mathfrak{g} -modules for the defining relations in $G_0(\mathfrak{g})$. For the next lemma see also Remark 2.12 of [BN1].

Lemma A.1 *The weight systems associated to a Lie superalgebra \mathfrak{g} with a fixed Casimir element ω induce a morphism of abelian groups*

$$W_{\mathfrak{g},\omega,\cdot} : G_0(\mathfrak{g}) \longrightarrow A^*, \quad [\rho] \mapsto W_{\mathfrak{g},\omega,\rho}$$

Proof: Let $M' \subset M$ be finite dimensional $\mathbb{Z}/(2)$ -graded vector-spaces and let $f : M \longrightarrow M$ be a linear map satisfying $f(M') \subseteq M'$. Denote the restriction of f to M' by f' and the induced endomorphism of $M'' := M/M'$ by f'' . Then the supertrace satisfies $\text{str}(f') + \text{str}(f'') = \text{str}(f)$. This completes the proof because $W_{\mathfrak{g},\omega,\rho}(D)$ can be described as the supertrace of $\rho(a_D)$ where a_D is an element of the universal enveloping algebra $U(\mathfrak{g})$ depending only on D and ω but not on ρ . \square

Now we can generalize Proposition 8.1 to semisimple Lie superalgebras over fields of characteristic 0.

Proposition A.1 *Let the ground field R be an arbitrary field of characteristic 0. For all $d \geq 8$ there exists an element $P_d \in P(A)_d$ with $\overline{\Gamma}(P_d) \neq 0$ and the following property: Let \mathfrak{g} be a finite direct sum of Lie superalgebras from Part II of Table (32) and let ω be a Casimir element for \mathfrak{g} . Then for all finite-dimensional representations ρ of \mathfrak{g} we have $W_{\mathfrak{g},\omega,\rho}(P_d) = 0$.*

Proof: First assume that R is algebraically closed of characteristic 0. Let $\mathfrak{g} = \bigoplus_{i=0}^k \mathfrak{g}_i$ be a direct sum of simple Lie superalgebras as in the proposition with Casimir element $\omega = \bigoplus_i \omega_i$. The following map is a well-defined morphism of abelian groups:

$$\pi_k : G_0(\mathfrak{g}_1) \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} G_0(\mathfrak{g}_k) \longrightarrow G_0(\mathfrak{g}) , \quad [\rho_1] \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} [\rho_k] \mapsto [\rho_1 \otimes \dots \otimes \rho_k].$$

In Theorem 8 of [Kac] simple \mathfrak{g}_i -modules are classified by highest weights. This classification implies a classification of the simple \mathfrak{g} -modules by highest weights. If we choose generators for each \mathfrak{g}_i as in Theorem 8 of [Kac] and v_i are highest weight vectors with respect to the chosen generators for \mathfrak{g}_i with weight α_i , then $v_1 \otimes \dots \otimes v_k$ is a highest weight vector for \mathfrak{g} with weight $\bigoplus_i \alpha_i$ because Cartan subalgebras are by definition of degree $\bar{0}$. Using a suitable order on the set of weights we can prove by induction that π_k is surjective.⁸

Let $m_k : (A^*)^{\otimes k} \longrightarrow A^*$ denote the k -fold product of weight systems. Exercise 6.33 of [BN1] holds also true for Lie superalgebras. Using this we see that the following diagram commutes:

$$\begin{array}{ccc} G_0(\mathfrak{g}_1) \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} G_0(\mathfrak{g}_k) & \xrightarrow{\pi_k} & G_0(\mathfrak{g}) \\ & \searrow & \swarrow \\ m_k \circ (W_{\mathfrak{g}_1, \omega_1, \cdot} \otimes \dots \otimes W_{\mathfrak{g}_k, \omega_k, \cdot}) & & W_{\mathfrak{g}, \omega, \cdot} \\ & \searrow & \swarrow \\ & A^* & \end{array}$$

Now let ρ be some finite-dimensional representation of \mathfrak{g} . We choose an element $\sum_i n_i [\rho_{i,1}] \otimes \dots \otimes [\rho_{i,k}] \in \pi_k^{-1}([\rho])$. The elements $P_d \in A_d$ from Proposition 8.1 satisfy $W_{\mathfrak{g}_j, \omega_j, \rho_{i,j}}(P_d) = 0$ for all i, j . This implies

$$W_{\mathfrak{g}, \omega, \rho}(P_d) = \sum_i n_i (W_{\mathfrak{g}_1, \omega_1, \rho_{i,1}} \cdot \dots \cdot W_{\mathfrak{g}_k, \omega_k, \rho_{i,k}})(P_d) = 0,$$

because P_d is primitive.

The structure of $G_0(\mathfrak{g})$ may change when the field R is not algebraically closed, but a simple argument will allow us to reduce this case to the algebraically closed one (compare also Exercise 6.32 of [BN1]). Let R be arbitrary of characteristic 0 and let \bar{R} be the algebraic closure of R . The Lie algebra $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ with Casimir $\omega = \bigoplus_i \omega_i$

⁸The theorem of Jordan-Hölder implies that π_k is an isomorphism but we will not use this.

and A are now defined over R and $\overline{R} \otimes \mathfrak{g}$ ($\overline{R} \otimes A$) denotes the extension of scalars from R to \overline{R} . Then the following diagram commutes for all R -linear projections $p : \overline{R} \longrightarrow R$.

$$\begin{array}{ccc}
G_0(\mathfrak{g}) & \xrightarrow{[\rho] \mapsto [id_{\overline{R}} \otimes \rho]} & G_0(\overline{R} \otimes \mathfrak{g}) \\
\downarrow W_{\mathfrak{g}, \omega, \cdot} & \searrow id_{\overline{R}} \otimes W_{\mathfrak{g}, \omega, \cdot} & \downarrow W_{\overline{R} \otimes \mathfrak{g}, \omega, \cdot} \\
& \overline{R} \otimes \text{Hom}_R(A, R) & \\
& \searrow \cong & \\
\text{Hom}_R(A, R) & \xleftarrow{p \circ (\varphi|_A) \leftarrow \varphi} & \text{Hom}_{\overline{R}}(\overline{R} \otimes A, \overline{R})
\end{array}$$

This completes the proof because the element P_d of the proof of Proposition 8.1 is defined over \mathbb{Q} . \square

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